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Exam One, MTH 213 , Fall 2021

Ayman Badawi

(Stop working at 13:00 pm/ submit your solution by 13:12 pm) $\overline{\quad 28 \quad}$

QUESTION 1. (9 points)(SHOW THE WORK)

(i) Use the 4-method and prove that $\sqrt{35}$ is irrational. [Hint: you may start by assuming $\sqrt{35} = a/b$ where a, b are odd integers and $gcd(a, b) = 1$].

(ii) By contradiction, show that $\sqrt{5} + \sqrt{7}$ is irrational. [Hint: you may use (i) above]

(iii) Assume that m, n are **POSITIVE** integers such that $m = n^2$. Use contradiction and prove that it is impossible that $m + 2 = k^2$ for some positive integer k.

QUESTION 2. (SHOW THE WORK)(4 points)

(i) Find 3 $(mod\, 8)$

(ii) Find -14 $(mod\, 23)$

QUESTION 3. (SHOW THE WORK)(3 points) Solve $12x = 8$ over planet Z_{20}.

QUESTION 4. (SHOW THE WORK)(3 points) Solve $7x = 5$ over planet Z_{10}

QUESTION 5. (SHOW THE WORK)(6 points) Let X be the number of students in class A, where $0 < X < 100$. Given $X\ (mod\ \ 7) = 5$, $X\ (mod\ \ 9) = 8$, and $X\ (mod\ \ 4) = 2$. Find X.

QUESTION 6. (SHOW THE WORK)(3 points) Use the division algorithm and find $gcd(204, 120)$

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Q.1) i) Deny. Hence $\sqrt{35}$ is rational

$\sqrt{35} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

$35 = \frac{a^2}{b^2}$

$\underset{\text{odd}}{35b^2} = \underset{\text{odd}}{a^2}$ we claim b and a are odd because if b is even $35 \times (\text{even})$ is even and $\gcd(a,b) \neq 1$

3

since b and a are odd, $b = 2k+1$, $a = 2m+1$ for some $k, m \in \mathbb{Z}$.

$35(2k+1)^2 = (2m+1)^2$

$35(4k^2+4k+1) = 4m^2+4m+1$

$35 \times 4k^2 + 35 \times 4k + 34 = 4m^2 + 4m$ $\boxed{\div 4 \text{ on both sides}}$

$\underbrace{35k^2 + 35k + \frac{17}{2}}_{\notin \mathbb{Z}} = \underbrace{m^2 + m}_{\text{Integer since } m \in \mathbb{Z}}$ $\Longleftarrow$ contradiction

✓

$\therefore$ due to the contradiction we conclude $\sqrt{35}$ is irrational.

Q1)ii) Deny. Hence $\sqrt{5}+\sqrt{7}$ is rational

$\sqrt{5}+\sqrt{7} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

$(\sqrt{5}+\sqrt{7})^2 = \frac{a^2}{b^2}$

3

$5 + \sqrt{35} + 7 = \frac{a^2}{b^2}$

$\underbrace{\sqrt{35}}_{\notin \mathbb{Q}} = \underbrace{\frac{a^2}{b^2} - \frac{12}{1}}_{\in \mathbb{Q}}$ $\Longleftarrow$ contradiction

LHS is irrational while the RHS is rational $\therefore$ due to this contradiction we conclude $\sqrt{5}+\sqrt{7}$ is irrational

(Q1) iii)

Deny. Hence it is possible that $m+2=k^2$ for some positive integer k

~~$m+2=k^2$~~

~~$m=k^2-2$~~

~~$n^2+2=k^2$~~

$n^2+2=k^2$

Assume n^2 is odd ∴ logically k^2 should also be odd (odd × odd = odd)

∵ odd + 2(even) = odd

~~since n,k are odd, n = 2a+1 for som a; k=2b+1 for~~

since n, k are odd, $n=2a+1$, $k=2b+1$ for $a,b \in \mathbb{Z}^+$

$n^2+2=k^2$

$(2a+1)^2+2=(2b+1)^2$

$4a^2+4a+3=4b^2+4b+①$

$4a^2+4a+2=4b^2+4b$ ÷4 on both sides

$\underbrace{a^2+a+\frac{1}{2}}_{\notin \mathbb{Z}^+} = \underbrace{b^2+b}_{\in \mathbb{Z}^+}$ ⇐ contradiction

∵ $\frac{1}{2}$ is not an integer

Hence due to one contradiction found, we conclude it's impossible that $m+2=k^2$ for some positive integer k

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if n is even ∴ logically k should be even too

Since n, k are even, $n=2h$, $k=2y$ for h,y $\mathbb{Z}^+$

$n^2+2=k^2$

$4h^2+2=4y^2$ $\boxed{÷4}$

$\underbrace{h^2+\frac{1}{2}}_{\notin \mathbb{Z}^+} = \underbrace{y^2}_{\in \mathbb{Z}^+}$ ← another contradiction

3

Good

(Q_2)i) $3(\bmod 8)$

$$\begin{array}{r} 0 \\ 8\overline{)\,3} \\ -0 \\ \hline 3 \end{array} = \underline{\underline{3}}$$

✓

(Q_2)ii) $-14(\bmod 23)$

$$\begin{array}{r} 0 \\ 23\overline{)\,14} \\ -0 \\ \hline 14 \end{array}$$

$-14(\bmod 23) = 23 - (14 \bmod 23))$

$= 23 - 14$

$= \underline{\underline{9}}$

✓

(Q_3) $12n = 8$ over planet Z_{20}

meaning: find all possible n, $0 \le n \le 19$ s.t. $12n(\bmod 20) = 8$

$a = 12$, $m = 20$, $b = 8$

$d = \gcd(a, m) = \gcd(12, 20) = \underline{\underline{4}}$

is $d|b$? => yes $4|8$ ∴ we conclude there are exactly 4 solutions

$12n(\bmod 20) = 8$

smallest $n = 4$

$\left\{ y = \frac{m}{d} = \frac{20}{4} = 5 \right.$

other solutions in the form $n + yk$ for some $k \in Z^+$

$x = \{4, 4+5(1), 4+5(2), 4+5(3)\}$

$= \{4, 9, 14, 19\}$

✓ 3

(Q_4) $7n = 5$ over Z_{10}

meaning: find all n, $0 \le n \le 9$, s.t. $7n(\bmod 10) = 5$

$a = 7$, $m = 10$, $b = 5$

$d = \gcd(a, m) = \gcd(7, 10) = 1$

Is $d|b$? => yes $1|5$ ∴ we conclude there is one exact solution

$7n(\bmod 10) = 5$ $\quad 35 \bmod 10 = 5$

$\underline{\underline{n = 5}}$

✓ 2

~~Q4) iii)~~

Q5) $x \pmod 7 = 5$

$x \pmod 9 = 8$

$x \pmod 4 = 2$

- $a_1 = 5$ $a_2 = 8$ $a_3 = 2$
- $m_1 = 7$ $m_2 = 9$ $m_3 = 4$

gcd (between every m_i's) = 1 ∴ CRT can be applied

- $m = 7 \times 9 \times 4 = 252$
- $n_1 = \frac{m}{m_1} = \underline{\underline{36}}$ • $n_2 = \frac{m}{m_2} = \underline{\underline{28}}$ • $n_3 = \frac{m}{m_3} = \underline{\underline{63}}$
- $n_1^{-1} (\bmod m_1) = 36^{-1} (\bmod 7)$

$36 \times \overset{36^{-1}}{x} (\bmod 7) = 1$

$\underline{\underline{x = 1}}$ ✓

- $n_2^{-1} (\bmod m_2) = 28^{-1} (\bmod 9)$

$28 \times \overset{28^{-1}}{x} (\bmod 9) = 1$

$\underline{\underline{x = 1}}$ ✓

- $n_3^{-1} (\bmod m_3) = 63^{-1} (\bmod 4)$

$63 \times x (\bmod 4) = 1$

$\underline{\underline{x = 3}}$ ✓

6

$$X = \left[\sum_{i=0}^{3} a_i n_i n_i^{-1}\right] \bmod m$$

$= [5 \times 36 \times 1 + 8 \times 28 \times 1 + 2 \times 63 \times 3] \bmod 252$ ✓

$= 782 (\bmod 252) = \underline{\underline{26}}$

$$\begin{array}{r|l} & 3 \\ 252 & 782 \\ & -756 \\ \hline & 26 \end{array}$$

Q6) gcd(204,120)

- $\begin{array}{r} 1 \\ 120\overline{)\,204} \\ -120 \\ \hline 84 \end{array}$

- $\begin{array}{r} 1 \\ 84\overline{)\,120} \\ -84 \\ \hline 36 \end{array}$

- $\begin{array}{r} 2 \\ 36\overline{)\,84} \\ -72 \\ \hline 12 \end{array}$

- $\begin{array}{r} 3 \\ 12\overline{)\,36} \\ -36 \\ \hline \underline{\underline{0}} \end{array}$

~~12~~ gcd (204,120) $= \underline{\underline{12}}$

✓ 3